2018

(October)

MATHEMATICS

(Honours)

(Elementary Number Theory and Advanced Algebra)

(GHS–51)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

Answer Elementary Number Theory and Advanced Algebra in two separate books.

UNIT—I

(Elementary Number Theory)

1. *(a)* State whether the following statements are True or False with brief justification (a, b, c, n denote integers) (any *five*) : 2×5=10

   (i) If $a|c$ and $b|c$, then $ab|c$.

   (ii) If $5\nmid(n-1)$, $5\nmid n$ and $5\nmid(n+1)$, then $5|n^2+1$.

(iii) $(a, bc) = 1 \Rightarrow (a, b) = 1$ and $(a, c) = 1$.

(iv) If $(a, b) = 1$, then $(a^2, b^2) = 1$.

(v) If $(a, b) = (a, c)$, then $[a, b] = [a, c]$.

(vi) $4 \nmid a^2 - 2$ for any integer a.

(b) Prove that if $a | n$, then $2^a - 1 \mid 2^n - 1$. 3

(c) Find the remainder, when the sum $S = 1! + 2! + 3! + \cdots + 1000!$ is divided by 8. 2

2. (a) State and prove Wilson's theorem. 1+5=6

(b) Prove that $n^5 - n$ is divisible by 30; for every integer n. 4

(c) Find the remainder, when 3^{247} is divided by 17. 3

(d) Find the number of positive integers less than 3600 that are relatively prime to 3600. 2

UNIT—II

3. (a) Solve the linear congruence $12x \equiv 44 \pmod{59}$. 3

(b) Solve the following systems of linear congruence : 5

$$x \equiv 3 \pmod{11}$$
$$x \equiv 5 \pmod{19}$$
$$x \equiv 10 \pmod{29}$$

(c) For any real number x, prove that

$$[x] + [-x] = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ -1, & \text{otherwise} \end{cases}$$ 4

(d) If n is an odd integer, prove that $\phi(2n) = \phi(n)$. 3

4. (a) Define Möbius function $\mu(n)$. 1

(b) Evaluate : 2

$$\sum_{j=1}^{\infty} \mu(j!)$$

(c) Prove that

$$\prod_{d|n} d = n^{\frac{\tau(d)}{2}}$$ 4

(d) Define the arithmetic function $\tau(n)$ for positive integers n and show that it is a multiplicative function. 4

(e) Evaluate $\sigma(4752)$ and $\tau(4752)$. 4

UNIT—III

(Advanced Algebra)

5. (a) If G is a group and H a subgroup of index 2 in G, prove that H is a normal subgroup of G. 5

(b) Prove that the necessary and sufficient condition for a homomorphism f of a group G into a group G' with Kernel K to be an isomorphism of G into G' is that $K = \{e\}$, where e is the identity of G. 5

(c) Prove that every field is an integral domain. 5

6. (a) Show that the set $Z[i]$ of Gaussian integers (i.e. the set of complex numbers $a+ib$, where a and b are integers) forms a ring under ordinary addition and multiplication of complex numbers. Is it an integral domain? Is it a field? Justify your answer in each case. 4+1+1=6

(b) If R is the additive group of real numbers and R^+ the multiplicative group of positive real numbers, prove that the mapping $f : R \to R^+$ defined by $f(x) = e^x$ for all $x \in R$ is an isomorphism of R onto R^+. 5

(c) If R is a finite commutative ring with unity element, prove that every prime ideal of R is a maximum ideal of R. 4

(Continued)

UNIT—IV

7. (a) Prove that a commutative ring with unity, is a field if it has no proper ideals. 5

(b) If f is a homomorphism of a ring R into a ring R' with Kernel S, then prove that S is an ideal of R. 4

(c) Show that every ideal I of an integral domain R is of the form $I = Ra$ for some $a \in R$. 4

(d) Show that the polynomial $x^2 - 3$ is irreducible over the field of rational numbers. 2

8. (a) Define units, prime elements of a Euclidean ring and the unique factorization domain. 2+2+2=6

(b) Define the term 'associates' in a Euclidean domain. In $\mathbb{Z}_5$, are 2 and 3 associates? 3

(c) Prove that every prime element in an integral domain with unit element is irreducible. 3

(d) Prove that every field is a Euclidean ring. 3

(Turn Over)

UNIT—V

9. (a) Is $W = \{(x, 2x, 3x) : x \in \mathbb{R}\}$ a one-dimensional subspace of $\mathbb{R}^3$, where $\mathbb{R}$ is the field of real numbers? Justify your answer. 3

(b) If F is the field of real numbers, show that the vectors $(1, 1, 0, 0)$, $(0, 1, -1, 0)$, $(0, 0, 0, 3)$ in $F^{(4)}$ are linearly independent over F. 3

(c) Determine whether or not the following vectors form a basis of R^3 :

$(1, 1, 2)$, $(1, 2, 5)$, $(5, 3, 4)$ 3

(d) Prove that two finite dimensional vector spaces $V(F)$ and $U(F)$ over a field F are isomorphic if and only if $\dim U = \dim V$. 6

10. (a) Let T be a linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$? 6

(Continued)

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(a, b) = (a+b, a-b, b)$ be a linear transformation. Find the range, rank and nullity of T. 5

(c) Is the vector $(2, -5, 3)$ in the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, -3, 2)$, $(2, -4, -1)$, $(1, -5, 7)$? Justify your answer. 4

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